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FIXED POINT FOR MAPPINGS OF ASYMPTOTICALLY NONEXPANSIVE TYPE IN LEBESGUE SPACES WITH VARIABLE EXPONENTS

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Dedicated to the memory of Professors K. Goebel and W.A. Kirk
thanking for their great contributions
in Banach Spaces Theory and Metric Fixed Point Theory

Abstract. Assume that (Ω,Σ,μ) is a σ-finite measure space and $p\colon \Omega \to [1,\infty]$ a variable exponent. In the case of a purely atomic measure, we prove that the w-FPP for mappings of asymptotically nonexpansive type in the Nakano space $\ell^{p(k)}$, where $p(k)$ is a sequence in $[1,\infty]$, is equivalent to several geometric properties of the space, as weak normal structure, the w-FPP for nonexpansive mappings and the impossibility of containing isometrically $L^1([0,1])$. In the case of an arbitrary σ-finite measure, we prove that this characterization also holds for pointwise eventually nonexpansive mappings. To determine if the w-FPP for nonexpansive mappings and for mappings of asymptotically nonexpansive type are equivalent is a long standing open question [19]. According to our results, this is the case, at least, for pointwise eventually nonexpansive mappings in Lebesgue spaces with variable exponents.

1. Introduction

In 1965, F. Browder [4] proved the existence of a fixed point for every nonexpansive mapping defined from a closed convex and bounded subset of a Hilbert space X into itself. The same year, F. Browder [5] and D. Göhde [14] proved

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that the same is true whenever X is a uniformly convex Banach space and W.A. Kirk [16] obtained a further extension for X being a reflexive Banach space with normal structure. These results stated a bridge between the fixed point theory and the geometrical theory of Banach spaces which can be considered as the foundation of the modern metric fixed point theory. This theory has widely developed in the last 50 years, receiving a lot of very relevant contributions and a very precise guidance from W.A. Kirk and K. Goebel (see, for instance, [13], [20] and references therein). One of the main targets of this theory is to find more and more general geometric properties of the space X which still keep the validity of the Kirk's Theorem. In some other cases, the authors study the existence of a fixed point for some more general classes of mappings, in particular, for mappings satisfying any type of asymptotic non-expansiveness. The first results in this direction were given by W.A. Kirk and K. Goebel [12]. They introduce the class of *asymptotically nonexpansive* mappings, i.e. mappings which satisfy that $d(T^n x, T^n y) \leq k_n d(x, y)$, $\{k_n\}$ being a sequence which converges to 1. In [12], the authors proved that these mappings have a fixed point when it is defined on a closed convex bounded subset of a uniformly convex Banach spaces. Fifty years on, it is still unknown whether Kirk's Theorem for reflexive Banach space with normal structure can be extended to this class of mappings. In 1974, W.A. Kirk [17] substantially weakened the asymptotic non-expansiveness assumption on T by replacing it with a condition which may hold even if none of the iterates of T is Lipschitzian, and he proved a fixed point result for these mappings which extends the one in [12] for uniformly convex spaces whenever T has a continuous iterated. Some further classes of mappings, which have been considered in the literature about metric fixed point theory, as pointwise eventually nonexpansive mappings and pointwise asymptotically nonexpansive mappings, are particular cases of this class of mappings of asymptotically nonexpansive type. We will recall the definitions of these notions in Section 2, where we will sketch the historical background about existence of a fixed point for these classes of mappings under the geometrical assumptions which are common in metric fixed point theory.

Section 3 is dedicated to recall the definition of the Lebesgue spaces with variable exponents $L^{p(\cdot)}(\Omega)$, where (Ω, Σ, μ) is a σ-finite measure. The class of variable Lebesgue spaces arises as a generalization of classic Lebesgue spaces $L^p(\Omega)$, when the constant exponent is replaced with a variable exponent function. Variable Lebesgue spaces can be traced back in the literature to 1931 [27] and they lie within the scope of the more general class of modular function spaces, initially defined by H. Nakano [26] and studied by Orlicz and Musielak [25]. Since M. Růžička discovered that they constitute a natural functional setting for the mathematical model of electrorheological fluids [30], variable Lebesgue spaces have witnessed an explosive development in the analysis of their intrinsic structure.

In Section 4, we state the main results of this paper. Our starting point is a technical result (Lemma 4.1) that is taken from [31]. We want to remark the meaning of this lemma. In 1975, K. Goebel [11] (see also [15]) proved the bizarre behavior of any weakly compact convex T-invariant set C which is minimal under these conditions when T is a nonexpansive mapping: nominally, $\lim_n \|x_n - x\| = \operatorname{diam}(C)$ for any approximate fixed point sequence $\{x_n\}$ in C of T and every $x \in C$. This result has proved to be very fruitful in metric fixed point theory, providing an efficient tool to find several different geometrical conditions that assure the existence of a fixed point for nonexpansive mappings (see, for instance, the monograph [2]). It is still unknown if this result also holds for mappings of asymptotically nonexpansive type. (In fact, it is unknown whether there exists an approximate fixed point sequence for such mappings). However, the iterates of a nonexpansive mapping satisfy another "bizarre" property in a minimal convex weakly compact T-invariant set C: $\limsup_n \|T^n x - z\|$ is a constant independent of $x, z \in C$ [13, Property 11.3]. In [31], it is proved that this property is still satisfied by mappings of asymptotically nonexpansive type defined in an appropriated minimal set, giving us a counterpart of Goebel–Karlovitz Lemma that can be used to obtain some fixed point results for this class of mappings under several geometrical assumptions [9], [21], [22], [24], [29], [30].

We use this lemma to prove the main results of this paper: for a purely atomic σ-finite measure, we prove the existence of a fixed point for mappings of asymptotically nonexpansive type defined in a convex weakly compact subset of the Nakano space $\ell^{p(k)}$, where $p\colon \mathbb{N} \to [1, \infty]$ when $\limsup_k p(k) < \infty$ and $p^{-1}(\{+\infty\})$ contains finitely many atoms, this condition being equivalent to weak normal structure, the w-FPP for nonexpansive mapping and the impossibility of containing isometrically $L^1([0, 1])$. In the case of an arbitrary σ-finite measure, we do not know if a similar result is true, but we can prove that it holds, at least, for pointwise eventually nonexpansive mapping defined in a convex weakly compact subset of the variable Lebesgue space $L^{p(\cdot)}(\Omega)$.

We want to remark that most previous fixed point results for mappings of asymptotically nonexpansive type derive from geometrical conditions (uniform convexity, uniform normal structure, nearly uniform convexity) which imply reflexivity. However, our results hold for Nakano spaces which are not, in general reflexive, as Remark 4.6 shows.

On the other hand, it is a long standing open question to determine if the existence of a fixed point for nonexpansive mappings and the existence for mappings of asymptotically nonexpansive type become equivalent problems. According to our result, this is the case for pointwise eventually nonexpansive mappings in Lebesgue spaces with variable exponents.

2. Mappings of asymptotically nonexpasive type

Definition 2.1. Let (X,d) be a metric space. A mapping $T\colon X \to X$ is called *nonexpansive* if

$$d(Tx,Ty) \le d(x,y) \quad \text{for every } x,y \in X.$$

It is well known that Browder's Theorem [4] does not hold for arbitrary Banach spaces. Thus, we can distinguish two classes of Banach spaces: those for which Browder's theorem holds and those for which it fails. We will fix the notation:

Definition 2.2. Let X be a Banach space. A subset C of X is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping $T\colon C \to C$ has a fixed point. We say that X has the *fixed point property* (FPP) for nonexpansive mappings if every bounded, closed, convex subset C has this property and we say that X has the *weak fixed point property* (w-FPP) for nonexpansive mappings if every convex weakly compact subset C satisfies the fixed point property.

Analogously, if we replace the class of nonexpansive mappings by a more general class $\mathcal{F}$, we will say that X satisfies the FPP (w-FPP) for the class $\mathcal{F}$. Note that if the Banach space is reflexive, the FPP and the w-FPP are equivalent for any class of mappings.

For many years, it was an open problem if every Banach space satisfies the w-FPP for nonexpansive mappings. This problem was solved by D.E. Alspach [1] in 1981, proving that the"baker" mapping defined in a weakly compact convex subset of $L^1([0,1])$ is a fixed point free nonexpansive mapping. (Forty years on, this is "essentially" the only known example of a weakly compact convex set failing the FPP for nonexpansive mappings).

The following definition relaxes in a natural way the non-expansiveness assumption:

Definition 2.3. Let X be a Banach space and C a nonempty subset of X. A mapping $T\colon C \to C$ is said to be *eventually nonexpansive* if there exists $N \in \mathbb{N}$ such that, for every $n \ge N$,

$$\|T^n x - T^n y\| \le \|x-y\|, \quad \text{for every } x,y \in X.$$

It should be noted that an eventually nonexpansive mapping does not need to be nonexpansive, nor even continuous.

Example 2.4. Let $C = [0, 1]$ and $T : [0, 1] \to [0, 1]$ defined by $T(x) = 0$ if $x < 1$ and $T(1) = 1/2$. It is clear that T is discontinuous at $x = 1$ but $T^n \equiv 0$ for every $n \geq 2$.

Looking at this example, we could guess that the fixed point theory for eventually nonexpansive mappings should be quite different of the corresponding theory for nonexpansive mappings. However, noting that the mappings T^n and T^{n+1} commute, the equivalence between both theories is a direct consequence of the existence of a common fixed point for two nonexpansive commuting mappings as proved in [6]. Thus, we have the following result:

Theorem 2.5 ([18]). *Let X be a Banach space which satisfies the* w-FPP. *Then, every eventually nonexpansive mapping T defined from a weakly compact convex set C into C has a fixed point.*

As mentioned in the introduction, W.A. Kirk [17] introduced in 1974 a further asymptotic extension of non-expansiveness and proved the Browder's Theorem [5] in this setting. We recall his definition:

Definition 2.6. A mapping $T : C \to C$ is said to be *of asymptotically nonexpansive type* if for each $x \in C$,

$$\lim_{n\to\infty}\sup\{\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}\} \leq 0.$$

The following fixed point result appeared in [17]:

Theorem 2.7. *Let X be a uniformly convex Banach space, C a closed convex bounded subset of X and $T : C \to C$ a mapping of asymptotically nonexpansive type. Assume that an iterate T^N of T is continuous. Then T has a fixed point.*

The absence of a continuous iterate can yield to trivial examples of mappings of asymptotically nonexpansive type which are fixed point free.

Example 2.8. Define $T : [0, 1] \to [0, 1]$ by $T(x) = x/2$ if $0 < x \leq 1$ and $T(0) = 1$. It is clear that T is fixed point free and for each $x \in [0, 1]$ we have $T^n x \leq 2^{1-n}$. Thus, $T^n(x) \to 0$ as $n \to \infty$ for all $x \in [0, 1]$ and $\lim_n \|T^n x - T^n y\| = 0$, for all $x, y \in [0, 1]$.

To avoid such a kind of trivial examples, we will consider in the following that the mapping T satisfies a weaker continuity condition, nominally, for every $x \in C$ there exists an integer $N = N(x)$ such that T^N is continuous at x. It must be noted that, as Example 3.3 in [9] shows, this condition does not imply the existence of a continuous iterated of T. We will call ANET mappings the mappings of asymptotically nonexpansive type which satisfy this weak continuity assumption.

Definition 2.9. A mapping $T\colon C \to C$ of asymptotically nonexpansive type will be called an ANET *mapping* if for each $x \in C$, there exists $N = N(x)$ such that T^N is continuous at x.

This condition is satisfied, for instance for the following classes of mappings, placed between eventually nonexpansive mappings and mappings of asymptotically nonexpansive type.

Definition 2.10 ([21], see also [19]). A mapping $T\colon C \to C$ is said to be *pointwise eventually nonexpansive* if for every $x \in C$ there exists $N(x) \in \mathbb{N}$ such that, if $n \geq N(x)$,

$$\|T^n x - T^n y\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Definition 2.11 ([19]). A mapping $T\colon C \to C$ is said to be *pointwise asymptotically nonexpansive* if for each $x \in C$ there exist $N(x) \in \mathbb{N}$ and a real sequence $\alpha_n(x)$ such that, if $n \geq N(x)$,

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\| \quad \text{for all } y \in C,$$

where $\lim_n \alpha_n(x) = 1$.

The relevance of the continuity of an iterate at a point is clarified in the following lemma:

Lemma 2.12. *Let X be an arbitrary topological space, M a nonempty subset of X and T a mapping from M into X. Assume that there exists $x \in M$ such that $\lim_n T^n x = x$ and there exists $N \in \mathbb{N}$ such that T^N is continuous at x. Then, $Tx = x$.*

Proof. Since $T^n x \to x$ and T^N is continuous at x, we have $T^N x = x$. Thus $T^{nN+1}x = Tx$ for all $n \geq 1$, which implies that $Tx = x$. □

Having in mind this lemma and revising the proof of Theorem 2.7, it is easy to check that the continuity assumption on T^N can be removed if we assume that T is an ANET mapping.

Although it is not yet known if the w-FPP for nonexpansive mappings is equivalent to the w-FPP for ANET mappings, besides Theorem 2.7, some classical existence results of fixed points for nonexpansive mappings have been extended to ANET mappings. We will recall some of them and the geometrical conditions that are used.

Definition 2.13. A Banach space X is said to have *normal structure* if $\operatorname{diam}(A)/r(A) > 1$ for any closed convex bounded set with $\operatorname{diam}(A) > 0$, where $r(A)$ denotes the Chebyshev radius of A, i.e. $r(A) = \inf\{\sup\{\|x - y\| : y \in A\} : x \in A\}$. The space X is said to have *weak normal structure* if the same is satisfied by any convex weakly compact subset of X. Finally, X is said to have *uniform normal structure* if

$$\inf\left\{\frac{\operatorname{diam}(A)}{r(A)} : A \text{ closed convex bounded with } \operatorname{diam}(A) > 0\right\} > 1.$$

Theorem 2.14 ([22], [28], [29]). *Every Banach space with uniform normal structure has the* w-FPP *for* ANET *mappings. In fact, since every space with uniform normal structure is reflexive, it also satisfies the* FPP *for this class of mappings.*

Definition 2.15. Let X be a Banach space and ϕ a measure of noncompactness on X. The space X is said to be *nearly uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if A is a subset of the unit ball satisfying $\phi(A) > \varepsilon$, then $d(0, A) \leq 1 - \delta$. Equivalently, $\Delta_\phi(\varepsilon) > 0$ for every $\varepsilon > 0$ where

$$\begin{aligned}\Delta_\phi(\varepsilon) = \sup\{c > 0 :\, &\text{for any bounded convex } A \subset B(0,1)\\ &\text{with } \phi(A) \geq \varepsilon, \text{ then } d(0, A) \leq (1 - c)\}.\end{aligned}$$

Theorem 2.16 ([9]). *Every nearly uniformly convex Banach space satisfies the* FPP *for* ANET *mappings.*

Nearly uniformly convex space are reflexive and have normal structure, but they do not have, in general, uniform normal structure.

3. Variable Lebesgue spaces and Nakano spaces

Since Lebesgue spaces with variable exponents are a particular case of modular function spaces, we begin recalling some definitions and some properties of these spaces.

Definition 3.1. Let $\mathcal{X}$ be an arbitrary vector space.

(a) A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a *convex modular* if for $x, y \in \mathcal{X}$:
   (i) $\rho(x) = 0$ if and only if $x = 0$;
   (ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;
   (iii) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

(b) A modular ρ defines a corresponding modular space, i.e. the vector space $\mathcal{X}_\rho$ given by $\{x \in \mathcal{X} : \rho(x/\lambda) < \infty \text{ for some } \lambda > 0\}$.

Given a vector space $\mathcal{X}$ with a convex modular ρ, the formula

$$\|x\| = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1\right\} \quad \text{for } x \in \mathcal{X}_\rho,$$

defines a norm which is frequently called the *Luxemburg norm* and $\mathcal{X}_\rho$ endowed with this norm is a Banach space.

Assume that (Ω, Σ, μ) is a σ-finite measure space. Let $p\colon \Omega \to [1, +\infty]$ be a measurable function and consider the vector space $\mathcal{X}$ of all measurable functions $g\colon \Omega \to \mathbb{R}$. Define the modular

$$\rho(g) := \int_{\Omega_f} |g(t)|^{p(t)}\, d\mu + \operatorname*{ess\,sup}_{p^{-1}(\{+\infty\})} |g(t)|, \tag{3.1}$$

where $\Omega_f := \{t \in \Omega : p(t) < +\infty\}$.

The variable Lebesgue Space $L^{p(\cdot)}(\Omega)$ is defined as the modular space endowed with the Luxemburg norm associated to the modular ρ defined above. It is well-known that $L^{p(\cdot)}(\Omega)$ is a Banach function lattice whose geometry is strongly attached to the behaviour of the exponent function $p(\cdot)$. Note that Lebesgue spaces $L^p(\Omega)$ endowed with the standard $\|\cdot\|_p$ norm $(1 \le p \le +\infty)$ are particular examples of this construction just by considering the constant function $p(t) = p$ for all $t \in \Omega$.

Following the usual notation, given a measurable set $E \subset \Omega$, we define

$$p_-(E) := \operatorname*{ess\,inf}_{t\in E} p(t), \qquad p_+(E) := \operatorname*{ess\,sup}_{t\in E} p(t).$$

If $E = \Omega$ we just denote $p_- := p_-(\Omega)$ and $p_+ := p_+(\Omega)$.

A modular space $\mathcal{X}_\rho$ is said to satisfy the Δ_2-condition if there exists $M > 0$ such that $\rho(2f) \le M\rho(f)$ for every $f \in \mathcal{X}_\rho$. It is easy to prove that $L^{p(\cdot)}(\Omega)$ satisfies the Δ_2-condition if $p_+(\Omega_f) < \infty$ (see [7, Proposition 2.14]). Moreover, in this case $\rho(g) < +\infty$ for every $g \in L^{p(\cdot)}(\Omega)$.

DEFINITION 3.2. The growth function $\omega_\rho \colon [0,\infty) \to [0,\infty)$ of a modular ρ is defined as follows:

$$\omega_\rho(t) := \sup\left\{\frac{\rho(tx)}{\rho(x)} : 0 < \rho(x) < \infty\right\} \quad \text{for all } t \ge 0.$$

LEMMA 3.3 ([10]). *Let ρ be a convex modular satisfying the Δ_2-condition. Then the growth function ω_ρ has the following properties:*

(a) $\omega_\rho(t) < \infty$ *for every* $t \in [0,\infty)$.
(b) $\omega_\rho(t) = 0$ *if and only if* $t = 0$.
(c) $\omega_\rho \colon [0,\infty) \to [0,\infty)$ *is a convex, strictly increasing function. So, it is continuous.*

REMARK 3.4. From the above lemma it follows that $\rho(x) \le \omega_\rho(\|x\|_\rho)$ for every $x \in \mathcal{X}_\rho$. Indeed, for every $\alpha > \|x\|_\rho$,

$$\rho(x) = \rho\left(\frac{\alpha x}{\alpha}\right) \le \omega_\rho(\alpha)\rho\left(\frac{x}{\alpha}\right) \le \omega_\rho(\alpha).$$

Letting α go to $\|x\|_\rho$ and using the continuity of $\omega_\rho(\cdot)$ we obtain the wanted inequality. As a consequence the modular is bounded on any norm-bounded subset of $\mathcal{X}_\rho$.

The following properties relating the modular and the Luxemburg norm will be used through this paper.

LEMMA 3.5 ([8]). *Let (Ω, Σ, μ) be a σ-finite measure, $p\colon \Omega \to [1,+\infty]$ be an exponent function, such that $p_+(\Omega_f) < \infty$. Then:*

(a) *For* $g \in L^{p(\cdot)}(\Omega)$ *we have* (a1) *If* $a \geq 1$, $a\rho(g) \leq \rho(ag) \leq a^{p_+(\Omega_f)}\rho(g)$.

(a2) *If* $0 < a < 1$, $a^{p_+(\Omega_f)}\rho(f) \leq \rho(af) \leq a\rho(f)$.

(b) *Assume that* (g_n) *is a sequence in* $L^{p(\cdot)}(\Omega)$. *Then:*

(b1) $\lim_n \|g_n\| = 1$ *if and only if* $\lim_n \rho(g_n) = 1$.

(b2) $\lim_n \|g_n\| = 0$ *if and only if* $\lim_n \rho(g_n) = 0$.

Note that, from Lemma 3.5, it is clear that $\rho(f) = 1$ if and only if $\|f\| = 1$ under the assumption $p_+(\Omega_f) < \infty$. Thus, if $\|f\| \geq 1$, from (a2) we have $1 = \rho(f/\|f\|) \geq \rho(f)/\|f\|^{p_+(\Omega_f)}$ which implies $\rho(f) \leq \|f\|^{p_+(\Omega_f)}$. Analogously $\rho(f) \geq \|f\|^{p_+(\Omega_f)}$ if $\|f\| \leq 1$.

The following result is the modular counterpart of the uniform convexity of $L^{p(\cdot)}(\Omega)$ whenever $1 < p_- \leq p_+ < \infty$.

LEMMA 3.6. *Let* (Ω, Σ, μ) *be a* σ*-finite measure,* $p : \Omega \to [1, +\infty]$ *be an exponent function. Assume* $1 < p_- \leq p_+ < \infty$. *For any* $1 < b < 2^{1/p_+}$, $2 \geq \delta > 0$ *there exists* $\alpha > 0$ *depending on* δ, p^+ *and* p_- *such that, for any* $u, v \in L^{p(\cdot)}(\Omega)$ *such that* $\|u\| \leq b$, $\|v\| \leq b$, $\|u - v\| \geq \delta$ *we have*

$$\frac{\rho(u) + \rho(v)}{2} \geq \rho\left(\frac{u+v}{2}\right) + \alpha.$$

PROOF. We adapt the proof of [23, Theorem 3.3]. Denote

$$\varepsilon^{p_-} = \frac{1}{3}\left(\frac{\delta}{2}\right)^{p_+}$$

and let $m > 0$ be the minimum of the function

$$(\lambda, p) \to \frac{1}{2}\left(|\lambda + 1|^p + |\lambda - 1|^p\right) - |\lambda|^p \quad \text{for } (\lambda, p) \in \left[-\varepsilon^{-1}, \varepsilon^{-1}\right] \times [p_-, p_+].$$

Denote $s = (u + v)/2$, $t = (u - v)/2$, $S = \{x \in \Omega : |t(x)| \leq \varepsilon|s(x)|\}$; $T = \{x \in \Omega : |t(x)| > \varepsilon|s(x)|\}$. From Lemma 3.5 and the convexity of the modular we have $\rho(s) \leq 2$ and $\rho(t) \geq (\delta/2)^{p_+}$. We have

$$\int_S |t(x)|^{p(x)}\, dx \leq \varepsilon^{p_-} \int_S |s(x)|^{p(x)}\, dx \leq 2\varepsilon^{p_-}.$$

Considering $\lambda = |s(x)/t(x)|$ we have

$$\frac{1}{2}\left(|s(x) + t(x)|^{p(x)} + |s(x) - t(x)|^{p(x)}\right) \geq \left|s(x)|^{p(x)} + m|t(x)\right|^{p(x)}$$

for every $x \in T$. Consequently,

$$\begin{aligned}\frac{1}{2}(\rho(u) + \rho(v)) &\geq \rho\left(\frac{u+v}{2}\right) + m\int_T |t(x)|^{p(x)}\, dx \\ &\geq \rho\left(\frac{u+v}{2}\right) + m\rho\left(\frac{u-v}{2}\right) - 2m\varepsilon^{p_-} \\ &\geq \rho\left(\frac{u+v}{2}\right) + m\left(\frac{\delta}{2}\right)^{p_+} - 2m\varepsilon^{p_-} \geq \rho\left(\frac{u+v}{2}\right) + \alpha\end{aligned}$$

where $\alpha = m\varepsilon^{p_-}$. □

When the measure space (Ω, σ, μ) is purely atomic, the exponent function $p(\,\cdot\,)$ can be considered as a sequence $(p(k))_k \subset [1, +\infty)$. The corresponding space is denoted by $\ell^{p(k)}$ and it is usually known in the literature as a Nakano space [26] (also a Musielak–Orlicz space [25]).

The following lemmas will be essential to prove our main results in Section 4. From now on, for $p\colon \Omega \to [1, \infty]$, we will denote $F_1 = p^{-1}(1)$, $F_\infty = p^{-1}(\infty)$, $F = F_1 \cup F_\infty$ and $\Lambda_\gamma = \{t \in \Omega_f : p(t) \geq \gamma$ almost everywhere$\}$.

Lemma 3.7 ([8, Lemma 3.2]). *Let (Ω, Σ, μ) be a σ-finite measure space and assume that the exponent function $p(\,\cdot\,)$ verifies $1 < p(t) < \infty$ almost everywhere. Let $u, v \in L^{p(\cdot)}(\Omega)$. Assume that there exists a ρ-bounded sequence (x_n) in $L^{p(\cdot)}(\Omega)$ verifying*

$$\lim_n \int_\Omega \bigg(|x_n(t) - u(t)|^{p(t)} + |x_n(t) - v(t)|^{p(t)} - 2 \left| x_n(t) - \frac{u(t) + v(t)}{2} \right|^{p(t)} \bigg) d\mu = 0. \tag{3.2}$$

then $u = v$ almost everywhere.

We recall the definition of asymptotic radius and center that will be used in the following:

Definition 3.8. Let $\{x_n\}$ be a bounded sequence in a metric space X and C a subset of X. The asymptotic radius of $\{x_n\}$ with respect to C is defined by

$$r_a(C, \{x_n\}) = \inf \Big\{ \limsup_n d(x_n, x) : x \in C \Big\}.$$

The asymptotic center is defined by

$$AC(C, \{x_n\}) = \Big\{ x \in C : \limsup_n d(x_n, x) = r_a(C, \{x_n\}) \Big\}.$$

It is clear that the asymptotic center of a bounded sequence can be empty. However, the asymptotic center is convex weakly compact and nonempty as C is.

Lemma 3.9. *Let (Ω, Σ, μ) be a σ-finite measure space and $p\colon \Omega \to [1, \infty]$ a measurable function such that $p_+(\Omega_f) < \infty$, F is purely atomic and F_∞ contains finitely many atoms at most. Let C be a weakly compact convex subset of $L^{p(\cdot)}(\Omega)$. Assume that $\{f_n\}$ is a sequence in C and $K = AC(C, \{f_n\})$. Then, K is a compact set and $u \cdot 1_{\Omega \setminus F} = v \cdot 1_{\Omega \setminus F} \in K$ for every $u, v \in K$.*

Proof. Since the cardinal of F_∞ is finite we know that $L^{p(\cdot)}(F)$ has the Schur property (because it is isomorphic to ℓ_1). We assume, by multiplication, that $r_a(C, \{f_n\}) = 1$ and select two arbitrary $u, v \in K$. We have that

$$1 = \limsup_n \left\| f_n - \left(\frac{u + v}{2} \right) \right\| = \limsup_n \|f_n - u\| = \limsup_n \|f_n - v\|.$$

From the assumption $p_+(\Omega_f) < +\infty$ and Lemma 3.5 we infer that

$$\lim_n \rho(f_n - u) = \lim_n \rho(f_n - v) = \lim_n \rho\left(f_n - \frac{u+v}{2}\right) = 1$$

and consequently

$$\lim_n \left[\rho(f_n - u) + \rho(f_n - v) - 2\rho\left(f_n - \frac{u+v}{2}\right)\right] = 0. \tag{3.3}$$

Note that we can write $f_n = g_n + h_n$, where $h_n = f_n \cdot 1_F$ and $g_n = f_n \cdot 1_{\Omega\setminus F}$. If we denote by

$$\rho_0(g) := \int_{\Omega\setminus F} |g|^{p(t)} d\mu \quad \text{and} \quad \rho_F(g) := \rho(g) - \rho_0(g) \quad \text{for } g \in L^{p(\cdot)}(\Omega),$$

we have

$$\rho(f_n - u) = \rho_0(g_n - u) + \rho_F(h_n - u)$$

and a similar decomposition is obtained for $\rho(f_n - v)$ and $\rho(f_n - (u+v)/2)$.

Condition (3.3) is now translated to $A_1 + A_2 = 0$, where

$$A_1 := \limsup_n \left[\rho_F(h_n - u) + \rho_F(h_n - v) - 2\rho_F\left(h_n - \frac{u+v}{2}\right)\right],$$
$$A_2 := \lim_n \left[\rho_0(g_n - u) + \rho_0(g_n - v) - 2\rho_0\left(g_n - \frac{u+v}{2}\right)\right].$$

By convexity of the modular we have that both $A_1, A_2 \geq 0$, and so $A_1 = A_2 = 0$. Consequently

$$\lim_n \int_{\Omega\setminus F} \left(|g_n(t) - u(t)|^{p(t)} + |g_n(t) - v(t)|^{p(t)} - 2\left|g_n(t) - \frac{u(t)+v(t)}{2}\right|^{p(t)}\right) d\mu = 0. \tag{3.4}$$

Due to the assumptions, we have $\sup_n \rho_0(g_n) < +\infty$. Furthermore, $1 < p(t) < +\infty$ almost everywhere in $\Omega \setminus F$. Consequently, applying Lemma 3.7 for the set $\Omega \setminus F$, we deduce that $u \cdot 1_{\Omega\setminus F} = v \cdot 1_{\Omega\setminus F}$ e.c.t. $\Omega \setminus F$. Due to the arbitrariness of the vectors $u, v \in K$, we can deduce that there exists a fixed $f \in K$ such that $u \cdot 1_{\Omega\setminus F} = f \cdot 1_{\Omega\setminus F}$ for any element u in K. Thus, K has the form $\{f \cdot 1_{\Omega\setminus F} + u \cdot 1_F : u \in K\}$. Since $L^{p(\cdot)}(F)$ satisfies the Schur property and C is weakly compact we have that K is a compact set. □

4. The w-FPP for mappings of asymptotically nonexpansive type in variable Lebesgue spaces

Assume that C is a nonempty weakly compact convex subset of a Banach space X and $T\colon C \to C$ is an arbitrary mapping. For each $x \in C$, let us denote by $\omega(x)$ the weak cluster point set of the sequence $\{T^n(x)\}$. Denote by $\mathfrak{F}$ the collection formed by all closed convex nonempty subsets D of C which contain

$\omega(x)$ for every $x \in D$. Let $\mathfrak{F}$ be ordered by inclusion. Then, $C \in \mathfrak{F}$ and, for every chain $\{D_i : I \in I\}$ in $\mathfrak{F}$, we have that $\bigcap_{i \in I} D_i \in \mathfrak{F}$. By Zorn's lemma, we obtain a minimal set K in $\mathfrak{F}$.

Lemma 4.1 ([31]). *Let C be a weakly compact convex subset of a Banach space X, $T\colon C \to C$ a mapping of asymptotically nonexpansive type and K a closed convex nonempty subset of C such that the cluster point set $\omega(x)$ of the sequence $\{T^n x\}$ is contained in K for every $x \in K$ and the set K is minimal under these conditions. Then, there exists $\rho \geq 0$ such that*

$$\limsup_n \|T^n x - y\| = \rho \quad \text{for every } x, y \in K.$$

The following lemma will play a key role in the proof of our main theorems.

Lemma 4.2 ([9, Lemma 4.5]). *Let C be a weakly compact convex subset of a Banach space X, $T\colon C \to C$ a mapping of asymptotically nonexpansive type. Assume that there exists a closed convex nonempty subset H of C which satisfies*

(a) *For each $x \in H$, $\omega(x) \subset H$.*
(b) *For each $x \in H$, every subsequence of $\{T^n x : n \in \mathbb{N}\}$ has a further convergent subsequence.*

Then there exists $z \in H$ such that $\{T^n z\}$ is norm convergent to z.

We can state now our first main theorem. We will use the following lemma:

Lemma 4.3. *Let $\{x_n = x_n(k)\}$ be a weakly null sequence in $\ell^{p(k)}$ where $p\colon \mathbb{N} \to (1, +\infty)$ and $\limsup_k p(k) < \infty$. For $\gamma > 1$ denote $A_\gamma = \mathbb{N} \setminus \Lambda_\gamma$. Then, for every $\varepsilon > 0$, there exists $\gamma > 1$ such that $\limsup_n \rho\big(x_n \cdot 1_{A_\gamma}\big) \leq \varepsilon$.*

Proof. We will assume, by contradiction, that for a positive ε and for every $\gamma > 1$, $\limsup_n \rho\big(x_n \cdot 1_{A_\gamma}\big) > \varepsilon$. Since $\{x_n\}$ converges to 0 coordinate-wise, we can construct an increasing sequence of positive integers $\{n_k\}$ and a sequence $\{y_k\}$ in $\ell^{p(k)}$ such that $\rho(x_{n_k}) > \varepsilon$, $\rho(y_k) > \varepsilon$, $\lim_k \|x_{n_k} - y_k\| = 0$, $\operatorname{supp} y_k$ is finite and $\operatorname{supp} y_k < \operatorname{supp} y_{k+1}$. It is clear that $\{y_k\}$ is a weakly null sequence. By induction, we can construct a sequence $\gamma_k < (k+1)/k$, a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and three subsequences $\{m_i\}$, $\{a_k\}$, $\{b_k\}$ of the positive integers such that $a_1 = 1$, $a_{k+1} > b_k$ and

(1) $\rho(y_{n_k} \cdot 1_{A_{\gamma_k}}) > \varepsilon$.
(2) $\operatorname{supp}\big(y_{n_k} \cdot 1_{A_{\gamma_k}}\big) = \{m_{a_k} < \ldots < m_{b_k}\}$.

Indeed, choose an arbitrary $\gamma_1 < 2$. Our assumption implies that there exists $n_1 \in \mathbb{N}$ such that $\rho\big(y_{n_1} \cdot 1_{A_{\gamma_1}}\big) > \varepsilon$. Let $\operatorname{supp}\big(y_{n_1} \cdot 1_{A_{\gamma_1}}\big) = \{m_1 < \ldots < m_{b_1}\}$. Assume that $\{y_{n_k}\}$, $\{\gamma_k\}$, $\{m_i\}$, $\{a_k\}$ and $\{b_k\}$ have been constructed for $k = 1, \ldots, h$. Choose $\gamma_{h+1} < (h+2)/(h+1)$. There exists n_{h+1} such that $\rho\big(y_{n_{h+1}} \cdot$

$1_{A_{\gamma_{h+1}}}\big) > \varepsilon$. Let $\operatorname{supp}\big(y_{n_{h+1}} \cdot 1_{A_{\gamma_{h+1}}}\big) = \big\{m_{a_{h+1}} < \ldots < m_{b_{h+1}}\big\}$. We have $a_{h+1} > b_h$ because $\operatorname{supp} y_{n_h} < \operatorname{supp} y_{n_{h+1}}$. Note that the sequence $p(m_i)$ converges to 1 because for every $k \in \mathbb{N}$ we have that $p(m_i) \in A_{\gamma_k}$ and so $p(m_i) < (k+1)/k$ except for finitely many i. Thus, $z_k =: \big\{y_{n_k} \cdot 1_{A_{\gamma_k}}\big\}$ is a sequence in $\ell^{p(m_i)}$ which satisfies $\rho(z_k) > \varepsilon$. It is easy to check that $\{z_k\}$ is weakly null. Indeed, since $p^+ < \infty$, is well known [7, Theorem 2.80] that the dual of $\ell^{p(m_i)}$ is $\ell^{q(m_i)}$, where $q(\cdot)$ is the conjugated exponent of $p(\cdot)$. Choose $u = u(m_i) \in \ell^{q(m_i)}$ and denote v the sequence in $\ell^{q(k)}$ defined by $v(m_i) = u(m_i)$ and $v(k) = 0$ if $k \notin \{m_i : i \in \mathbb{N}\}$. Then $v(y_n) \to 0$ which implies that $u(z_k) = v(y_{n_k}) \to 0$. However, $\ell^{p(m_i)}$ satifies the Schur Property (see [3, Theorem 5.1 and Corollary 7.4]) which implies that $\rho(z_k) \to 0$ contradicting $\rho(z_k) > \varepsilon$. □

Theorem 4.4. *Let $p\colon \mathbb{N} \to [1, +\infty]$. The following conditions are all equivalent:*

(a) *$\ell^{p(k)}$ has weak normal structure.*
(b) *$\ell^{p(k)}$ satisfies the* w-FPP *for nonexpansive mappings.*
(c) *$\ell^{p(k)}$ does not contain isometrically $L^1[0,1]$.*
(d) *$\limsup_k p(k) < \infty$, and $p^{-1}(\{+\infty\})$ contains finitely many atoms.*
(e) *$\ell^{p(k)}$ satisfies the* w-FPP *for mappings of asymptotically nonexpansive type.*

Proof. By [8, Theorem 3.3] conditions (a)–(d) are all equivalent. Since obviously (e) implies (b) we only need to prove that (d) implies (e). Let K be a minimal convex set which satisfies $\omega(x) \subset K$ for every $x \in K$. By normalization we assume that $\limsup \|T^n x - y\| = 1$ for every $x, y \in K$. We will prove that condition (b) in Lemma 4.2 is satisfied and so, the existence of a fixed point for T will be a consequence of Lemma 2.12. Indeed, if $\{T^{n_k}x\}$ does not have any convergent subsequence, we can assume that there exists $d > 0$ such that $d \le \|T^{n_k}x - T^{n_j}x\|$ for every $k, j \in \mathbb{N}$ and $\{T^{n_k}x\}$ is weakly convergent, say to $w \in K$. From Lemma 4.3 aplied to the sequence $\{T^{n_k}x - w\}$, there exists $\gamma > 1$ such that $\big\|(T^{n_k}x - T^{n_j}x) \cdot 1_{\mathbb{N}\setminus(F\cup\Lambda_\gamma)}\big\| < d/3$. Since $\ell^{p(\cdot)} \cdot 1_F$ satisfies the Schur property, we can assume $\big\|\big(T^{n_k}x - T^{n_j}x\big) \cdot 1_F\big\| < d/3$ and so $\big\|\big(T^{n_k}x - T^{n_j}x\big) \cdot 1_{\Lambda_\gamma}\big\| > d/3$. Finally, we can apply Lemma 3.6, for $\delta = d/3$, to obtain $\alpha > 0$ such that

$$\frac{\rho(x) + \rho(y)}{2} \ge \rho\left(\frac{x+y}{2}\right) + \alpha$$

for any $x, y \in \ell^{p(\cdot)} \cdot 1_{\Lambda_\gamma}$ such that $\|x\| < b$, $\|y\| \le b$, $\|x - y\| \ge d/3$. Since $p_+(\mathbb{N}_f) < \infty$, $\ell^{p(\cdot)} \cdot 1_{N_f}$ satisfies the Δ_2-condition and from Lemma 3.3 and Remark 3.4, there exists β such that $\omega(1+\beta) \le 1 + \alpha/3$ and so $\rho(x) \le 1 + \alpha/3$ whenever $\|x\| \le 1 + \beta$. Since, $\limsup_n \|T^n x - x\| = 1$ we can choose n_0 such that $\|T^n x - x\| \le 1 + \beta/2$ and $\sup\big\{\|T^n x - T^n u\| - \|x - u\| : u \in C\big\} \le \beta/2$ for every

$n \geq n_0$. Fixed $n \geq n_0$, we choose k_0 large enough such that $n_k \geq n + n_0$ for $k \geq k_0$. Thus, for $k \geq k_0$ we have

$$\|T^n x - T^{n_k} x\| \leq \|x - T^{n_k - n} x\| + \beta/2 \leq 1 + \beta$$

which implies $\rho(T^n x - T^{n_k} x) \leq 1 + \alpha/3$. Thus we have

$$\begin{aligned} 2\,\rho\bigg(\frac{T^{n_k}x + T^{n_{k+1}}x}{2} - T^n x\bigg) &= 2\,\rho\bigg(\bigg(\frac{T^{n_k}x + T^{n_{k+1}}x}{2} - T^n x\bigg)\cdot 1_{\Lambda_\gamma}\bigg) \\ &\quad + 2\rho\bigg(\bigg(\frac{T^{n_k}x + T^{n_{k+1}}x}{2} - T^n x\bigg)\cdot 1_{\mathbb{N}\setminus\Lambda_\gamma}\bigg) \\ &\leq \rho\big((T^{n_k}x - T^n x)\cdot 1_{\Lambda_\gamma}\big) + \rho\big((T^{n_{k+1}}x - T^n x)\cdot 1_{\Lambda_\gamma}\big) - \alpha \\ &\quad + \rho\big((T^{n_k}x - T^n x)\cdot 1_{\mathbb{N}\setminus\Lambda_\gamma}\big) + \rho\big((T^{n_{k+1}}x - T^n x)\cdot 1_{\mathbb{N}\setminus\Lambda_\gamma}\big) \\ &= \rho\big(T^{n_k}x - T^n x\big) + \rho\big(T^{n_{k+1}}x - T^n x\big) - \alpha < 2 - \alpha/3 \end{aligned}$$

which implies that there exists $c < 1$ $(c = (1 - \alpha/6)^{1/p_+(\mathbb{N}_f)})$ such that

$$\bigg\|\frac{T^{n_k}x + T^{n_{k+1}}x}{2} - T^n x\bigg\| < c$$

for every $n \geq n_0$. Taking $\liminf$ as $k \to \infty$ we obtain $\|w - T^n x\| \leq c$ for every $n \geq n_0$. Thus, we reach the contradiction $\limsup_n \|T^n x - w\| < 1$. □

We do not know if the above result is true for an arbitrary σ-finite measure, but it holds, al least, for pointwise eventually nonexpansive mappings:

Theorem 4.5. *Let (Ω, Σ, μ) be a σ-finite measure space and $p\colon \Omega \to [1, +\infty]$ be a measurable function. The following conditions are all equivalent*:

(a) *$L^{p(\cdot)}(\Omega)$ satisfies the weak normal structure.*
(b) *$L^{p(\cdot)}(\Omega)$ satisfies the* w-FPP *for nonexpansive mappings.*
(c) *$L^{p(\cdot)}(\Omega)$ does not contain isometrically $L^1[0,1]$.*
(d) *$p_+(\Omega_f) < +\infty$, $p^{-1}(\{+\infty\})$ contains finitely many atoms at most and every measurable atomless subset of $p^{-1}(\{1, +\infty\})$ is negligible.*
(e) *$L^{p(\cdot)}(\Omega)$ satisfies the* w-FPP *for pointwise eventually nonexpansive mappings.*

Proof. We only need to prove that (d) implies (e) as in Theorem 4.5. By Lemma 3.9, any sequence lying in a weakly compact convex set C has a compact asymptotic center $AC(C, \{x_n\})$. Thus, the result is a consequence of [9, Theorem 5.6]. □

Remark 4.6. Usual assumptions to obtain a fixed point for ANET mappings (uniform convexity, uniform normal structure, nearly uniform normal structure) also imply reflexivity of X. However, in the case of the Nakano spaces $\ell^{p(k)}$, $p\colon \mathbb{N} \to (1, \infty)$, it is known that they are not reflexive if $p_- = \liminf_n p(n) = 1$.

If the following conjecture were true, it would contain Theorems 4.4 and 4.5 as particular cases.

CONJECTURE 4.7. Let (Ω, Σ, μ) be a σ-finite measure space and $p\colon \Omega \to [1, +\infty]$ be a measurable function. The following conditions are all equivalent:

(a) $L^{p(\cdot)}(\Omega)$ satisfies the weak normal structure.
(b) $L^{p(\cdot)}(\Omega)$ satisfies the w-FPP for nonexpansive mappings.
(c) $L^{p(\cdot)}(\Omega)$ does not contain isometrically $L^1[0,1]$.
(d) $p_+(\Omega_f) < +\infty$, F is purely atomic and F_∞ contains finitely many atoms at most.
(e) $L^{p(\cdot)}(\Omega)$ satisfies the w-FPP for mappings of asymptotically nonexpansive type.

REFERENCES

[1] D.E. ALSPACH, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 423–424.

[2] J. M. AYERBE, T. DOMÍNGUEZ BENAVIDES AND G. LÓPEZ ACEDO, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory: Advances and Applications, vol. 99, Birkäuser, Basel, 1997.

[3] A. BARRERA-CUEVAS AND M. JAPÓN, *New families of nonreflexive Banach spaces with the fixed point property*, J. Math. Anal. Appl. **425** (2015), no. 1, 349–363.

[4] F.E. BROWDER, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Nat. Acad. Sci. USA **43** (1965), 1272–1276.

[5] F.E. BROWDER, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54** (1965), 1041–1044.

[6] R.E. BRUCK, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific J. Math. **53** (1974), 59–71.

[7] D.V. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces*, Birkhäuser, Basel, 2013.

[8] T. DOMÍNGUEZ BENAVIDES AND M. JAPÓN, *Fixed point properties and reflexivity in variable Lebesgue spaces*, J. Funct. Anal. **280** (2021), no. 6, paper no. 108896.

[9] T. DOMÍNGUEZ BENAVIDES AND P. LORENZO, *Fixed points for mappings of asymptotically nonexpansive type*, Fixed Point Theory **24** (2023), no. 2, 569–582,

[10] T. DOMÍNGUEZ BENAVIDES, M.A. KHAMSI AND S. SAMADI, *Asymptotically nonexpansive mappings in modular function spaces*, J. Math. Anal. Appl. **265** (2002), no. 2, 249–263.

[11] K. GOEBEL, *On the structure of minimal invariant sets for nonexpansive mappings*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **29** (1975), 73–77.

[12] K. GOEBEL AND W.A. KIRK, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[13] K. GOEBEL AND W.A. KIRK, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[14] D. GÖHDE, *Zum Prinzip der kontraktiven Abbildung*, Math. Nach. **30** (1965), 251–258.

[15] L.A. KARLOVITZ, *Existence of fixed points of nonexpansive mappings in a space without normal structure*, Pacific J. Math. **66** (1976), 153–159.

[16] W. KIRK, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72**, 1004–6.

[17] W.A. KIRK, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 339–346.

[18] W.A. KIRK, *The fixed point property and mappings which are eventually nonexpansive*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 141–147.

[19] W.A. KIRK, *Remarks on nonexpansive mappings and related asymptotic conditions*, J. Nonlinear Convex Anal. **18** (2017), no. 1, 1–15.

[20] W.A. KIRK AND B. SIMS (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.

[21] W.A. KIRK AND H. K. XU, *Asymptotic pointwise contractions*, Nonlinear Anal. **69** (2008), 4706–4712.

[22] G. LI AND B. SIMS, *Fixed point theorems for mappings of asymptotically nonexpansive type*, Nonlinear Anal. **50** (2002), 1085–1091.

[23] J. LUKEŠ, L. PICK AND D. POKORNÝ, *On geometric properties of the spaces* $L^{p(x)}$, Rev. Mat. Complut. **24** (2011), no. 1, 115–130.

[24] R. MALIK AND S. RAJESH, *Fixed points of asymptotically nonexpansive type mappings*, Adv. Oper. Theory **8** (2023), no. 9, DOI: 10.1007/s43036-022-00235-9.

[25] J. MUSIELAK, W. ORLICZ, *On modular spaces*, Studia Math. **18** (1959), 591–597.

[26] H. NAKANO, *Modulared Semi-ordered Linear Spaces*, Maruzen Co., Tokyo, 1950.

[27] W. ORLICZ, *Über konjugierte Exponentenfolgen*, Studia Math. **3**, (1931), 200–212.

[28] M. RADHAKRISHNAN AND S. RAJESH, *Existence of fixed points for pointwise eventually asymptotically nonexpansive mappings*, Appl. Gen. Topol. **20**, no. 1 (2019), 119–133.

[29] S. RAJESH, *On existence of fixed points for pointwise eventually nonexpansive mappings*, J. Fixed Point Theory Appl. **19** (2017), no. 3, 2177–2184.

[30] M. RŮŽIČKA, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, vol. 1748, Springer, Berlin, 2000.

[31] H.K. XU, *Existence and convergence of fixed points for mappings of asymptotically nonexpansive type*, Nonlinear Anal. **16** (1991), 1139–1146.

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